Eq. (9) into Eq. (8) and using Eq. (6),

$$\frac{d^2g}{dxdy} = \frac{\partial^2 g}{\partial x \partial y} + \left(\frac{dU}{dy}\right)^T R \frac{dU}{dx} + \left(\frac{\partial Z}{\partial y}\right)^T \frac{dU}{dx} + \left(\frac{\partial Z}{\partial x}\right)^T \frac{dU}{dy}$$

$$+ \Lambda^T \left[\frac{d^2F}{dxdy} - \frac{d^2K}{dxdy} U - \frac{dK}{dx}\frac{dU}{dy} - \frac{dK}{dy}\frac{dU}{dx}\right] \tag{10}$$

The expression in Eq. (10) for the second derivatives requires solution of systems like Eqs. (4) and (6) for the first derivatives of the displacement vector and for Λ, respectively. For n design variables, this requires $n+1$ solutions. This is about half of what is required by Haug's approach. When there are m constraints, Haug's approach requires $n(m+1)$ solutions because one of his adjoint vectors depends also on the constraints. The present approach requires only $n+m$ solutions. This is almost always better than the use of Eq. (8), with $n+n(n+1)/2$ required solutions.

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Structure of Thermal Radiation Field in an Optically Thick Limit

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Nomenclature

G = incident radiation
I = radiation intensity
I_w = Planck function at wall temperature
Q = heat flux in radial direction
r = radial direction
R = radius
T = temperature
β = extinction coefficient
ν = frequency
ω = scattering albedo
ϵ_w = wall emissivity
η = stretched coordinate

Subscripts

b = black body
c = centerline
D = diffusion approximation
i = inner region
o = outer region
w = wall
ν = spectral

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I. Introduction

IN the optically thick limit of thermal radiation transport, the mean free path for radiation becomes much smaller than the dimension of the medium, so that energy transfer at any position depends only on the conditions in the immediate vicinity. Consequently, the energy transfer can be approximated as a diffusion process, instead of being given as a solution of the more complicated integral equation. The diffusion approximation imposes a requirement that the radiation field be nearly isotropic, a condition that may be fulfilled in the interior of the gas but is violated at the wall boundary. This restriction was recognized by, among others,[1] Deissler,[2] who introduced a concept of jump in emissive power at the wall to patch up the inaccuracy of diffusion approximation in the wall region. The problem analyzed by Deissler was one of radiative equilibrium. The assumption of radiative equilibrium implies that the system is in steady state and the only mode of heat transfer is thermal radiation. For a system in radiative equilibrium, the divergence of heat flux is zero everywhere. Consequently, the heat flux is spatially constant throughout the radiation field (for a planar medium) so that the absorption and emission of radiation are in local equilibrium everywhere. We will call this a structureless radiation field since the incident radiation is given trivially in terms of the emissive power of the medium.

The present work investigates the transport of radiation in the optically thick limit when the temperature profile within the medium is specified and is continuous. It is emphasized that the medium being considered is not in radiative equilibrium, i.e., the divergence of heat flux is not zero anywhere in the radiation field. Consequently, the incident radiation is not identically equal to the local emissive power of the medium so that there is a structure to the radiation field. It is this structure that we wish to analyze for the case of large optical thickness. It is mentioned, in advance, that for the present problem the incident radiation turns out to be nonanalytic and is analyzed by using the singular perturbation technique. The temperature profile, however, is specified and is analytic. The nonanalytic temperature profiles for optically thick absorbing-emitting media in radiative equilibrium have been analyzed, using the singular perturbation techniques, by Emanuel for plane-parallel[3] and cylindrical geometries,[4] and by Lin and Chan[5] for a planar nongray medium.

II. General Formulation

Consider an axisymmetric medium that absorbs, emits, and isotropically scatters thermal radiation. Neglecting heat transfer in the axial direction (long L/D ratio), the following governing equation for the spectral incident radiation G_ν is obtained by applying the P_1 approximation to the radiation transport equation in cylindrical coordinates.[6]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\beta_\nu} r \frac{\partial G_\nu}{\partial r}\right) - 3(1-\omega_\nu)\beta_\nu G_\nu = -12\pi(1-\omega_\nu)\beta_\nu I_{b\nu} \tag{1}$$

For a diffusely reflecting wall, the boundary conditions are

$$\frac{\partial G_\nu}{\partial r} = 0, \qquad r = 0 \tag{2}$$

$$\frac{2}{3}\frac{(2-\epsilon_w)}{\epsilon_w}\frac{1}{\beta_\nu}\frac{\partial G_\nu}{\partial r}+G_\nu=4\pi I_{w\nu},\quad r=R \tag{3}$$

The local heat flux and its divergence are related to G_ν as follows.

$$Q_\nu=-\frac{1}{3\beta_\nu}\frac{\partial G_\nu}{\partial r} \tag{4}$$

$$\nabla . Q_\nu=4\pi(1-\omega_\nu)\beta_\nu\left(I_{b\nu}-\frac{G_\nu}{4\pi}\right) \tag{5}$$

Singular Behavior

The nature of singularity of Eq. (1) can be observed by noting that in the limit of β_ν approaching infinity, Eq. (1) has the following degenerate form.

$$G=4\pi I_b \tag{6}$$

In Eq. (6) and the remainder of this study, the subscript ν has been dropped for simplicity of representation. Equation (6) is the statement of local equilibrium between the absorption and emission; this is the generally accepted optically thick behavior. However, it is not a uniformly valid solution, as can be shown by transforming Eq. (1) along with its boundary conditions from r to y coordinate where

$$y=1-r/R \tag{7}$$

Define

$$\hat{G}=G-4\pi I_w \tag{8}$$

The transformed equation and boundary conditions are

$$\epsilon\frac{1}{(1-y)}\frac{\partial}{\partial y}\left[(1-y)\frac{\partial\hat{G}}{\partial y}\right]-\hat{G}=-4\pi(I_b-I_w) \tag{9}$$

$$\epsilon^{1/2}\gamma\frac{\partial\hat{G}}{\partial y}-\hat{G}=0,\quad y=0 \tag{10}$$

$$\frac{\partial\hat{G}}{\partial y}=0,\quad y=1 \tag{11}$$

where

$$\epsilon=\frac{1}{3(1-\omega)\beta^2R^2},\quad \epsilon<1 \tag{12}$$

$$\gamma=\frac{2}{\sqrt{3}}(1-\omega)^{1/2}\left(\frac{2-\epsilon_w}{\epsilon_w}\right) \tag{13}$$

In the foregoing equations, ϵ is a parameter that is inversely proportional to the square of the optical thickness. In the optically thick limit, $\beta\rightarrow\infty$ so that $\epsilon\rightarrow 0$, and the vanishing small parameter ϵ multiplies the highest derivative of the governing equation, Eq. (9)—a classical sign of singular behavior. Clearly, in the vicinity of the wall (i.e., $y=0$) $\hat{G}$ must vary in a fashion so as to have the first derivative with respect to y of $0(1/\epsilon^{1/2})$ and second derivative of $0(1/\epsilon)$. More formally, define a stretched variable η

$$\eta=y/\Gamma(\epsilon) \tag{14}$$

If Eqs. (9) and (10) were transformed from the y to the η coordinate according to Eq. (14), then a functional form for Γ that would allow retention of the highest derivative terms is the following.

$$\Gamma=\epsilon^{1/2} \tag{15}$$

Physically, Eq. (15) states that the thickness of the wall region in which the wall influences the radiation field is $\epsilon^{1/2}R$. This wall region is called the "inner region" as opposed to the "outer region" in which the wall effects are relatively unimportant. Inner and outer regions have different length scales characterizing them, the inner length scale is merely $\epsilon^{1/2}$ times its counterpart in the outer region. The objective of the next sections is to obtain solutions separately valid for the two regions and to appropriately match them. The solutions are obtained by the method of inner and outer expansions.[7]

III. Solution by Singular Perturbation Method

As explained in the last section, equations in the inner region are written in the η coordinate, and in the outer region in the y coordinate. The inner region equations are

$$\frac{1}{(1-\eta\epsilon^{1/2})}\frac{\partial}{\partial\eta}\left[(1-\eta\epsilon^{1/2})\frac{\partial\hat{G}}{\partial\eta}\right]-\hat{G}=-4\pi(I_b-I_w) \tag{16}$$

and

$$\gamma\frac{\partial\hat{G}}{\partial\eta}-\hat{G}=0,\quad \eta=0 \tag{17}$$

The outer region equations are

$$\frac{\epsilon}{(1-y)}\frac{\partial}{\partial y}\left[(1-y)\frac{\partial\hat{G}}{\partial y}\right]-\hat{G}=-4\pi(I_b-I_w) \tag{18}$$

and

$$\frac{\partial\hat{G}}{\partial y}=0,\quad y=1 \tag{19}$$

Equations (16) and (18) are second-order differential equations with only one boundary condition specified. The missing boundary conditions are to be derived through matching. Before proceeding further, it should be noted that (I_b-I_w) admits the following Taylor series expansion about $y=0$.

$$I_b-I_w=\left(\frac{\partial I_b}{\partial y}\right)_0 y+\left(\frac{\partial^2 I_b}{\partial y^2}\right)_0\frac{y^2}{2}+\left(\frac{\partial^3 I_b}{\partial y^3}\right)_0\frac{y^3}{6}+\ldots \tag{20}$$

Inner Region

For finite η but $\epsilon\rightarrow 0$,

$$\frac{1}{(1-\eta\epsilon^{1/2})}=1+\eta\epsilon^{1/2}+\eta^2\epsilon+0(\epsilon^{3/2}) \tag{21}$$

In terms of the inner variable η, Eq. (20) can be rewritten as

$$(I_b-I_w)=\left(\frac{\partial I_b}{\partial y}\right)_0\eta\epsilon^{1/2}+\frac{1}{2}\left(\frac{\partial^2 I_b}{\partial y^2}\right)_0\eta^2\epsilon+\frac{1}{6}\left(\frac{\partial^3 I_b}{\partial y^3}\right)_0\eta^3\epsilon^{3/2}+0(\epsilon^2) \tag{22}$$

With the help of Eqs. (21) and (22), Eq. (16) can be recast as

$$\frac{\partial^2\hat{G}_i}{\partial\eta^2}-[\epsilon^{1/2}+\eta\epsilon+\eta^2\epsilon^{3/2}+0(\epsilon^2)]\frac{\partial\hat{G}_i}{\partial\eta}-\hat{G}_i=-4\pi\left(\frac{\partial I_b}{\partial y}\right)_0\eta\epsilon^{1/2}-2\pi\left(\frac{\partial^2 I_b}{\partial y^2}\right)_0\eta^2\epsilon-\frac{2\pi}{3}\left(\frac{\partial^3 I_b}{\partial y^3}\right)_0\eta^3\epsilon^{3/2}+0(\epsilon^2) \tag{23}$$

In the foregoing equation, the subscript i refers to inner region. Let

$$\hat{G}_i(\epsilon,\eta) = \epsilon^{1/2}\hat{G}_{i1}(\eta) + \epsilon\hat{G}_{i2}(\eta) + \epsilon^{3/2}\hat{G}_{i3}(\eta) + O(\epsilon^2) \tag{24}$$

Substituting Eq. (24) into Eq. (23), the following governing equations for the first- and second-order inner solutions can be derived.

$$\frac{\partial^2 \hat{G}_{i1}}{\partial \eta^2} - \hat{G}_{i1} = -4\pi\left(\frac{\partial I_b}{\partial y}\right)_0 \eta \tag{25}$$

$$\frac{\partial^2 \hat{G}_{i2}}{\partial \eta^2} - \hat{G}_{i2} = \frac{\partial \hat{G}_{i1}}{\partial \eta} - 2\pi\left(\frac{\partial^2 I_b}{\partial y^2}\right)_0 \eta^2 \tag{26}$$

The boundary conditions at $\eta = 0$ for Eqs. (25) and (26) are

$$\gamma\frac{\partial \hat{G}_{i1}}{\partial \eta} - \hat{G}_{i1} = 0 \tag{27}$$

$$\gamma\frac{\partial \hat{G}_{i2}}{\partial \eta} - \hat{G}_{i2} = 0 \tag{28}$$

Before attempting to solve Eqs. (25) and (26), the missing boundary conditions must be derived. For this purpose, attention is directed to the expansion in the outer region.

Outer Region

In the outer region, the following general series expansion for $\hat{G}_o$ may be adopted (the subscript o denotes the outer region).

$$\hat{G}_o(\epsilon,y) = \hat{G}_{o1}(y) + \epsilon\hat{G}_{o2}(y) + \epsilon^2\hat{G}_{o3}(y) + O(\epsilon^3) \tag{29}$$

The following closed-form solutions exist for the first- and second-order outer terms which implicitly satisfy the conditions of vanishing derivatives at $y = 1$.

$$\hat{G}_{o1} = 4\pi(I_b - I_w) \tag{30}$$

$$\hat{G}_{o2} = \frac{1}{(1-y)}\frac{\partial}{\partial y}\left[(1-y)\frac{\partial \hat{G}_{o1}}{\partial y}\right] \tag{31}$$

Matching

In order to derive the missing boundary conditions for the inner solutions, the behavior of $\hat{G}_o$ in the limit of y approaching zero is investigated. From Eqs. (29-31)

$$\hat{G}_o = 4\pi(I_b - I_w) + \epsilon 4\pi\left(\frac{\partial^2 I_b}{\partial y^2} - \frac{1}{(1-y)}\frac{\partial I_b}{\partial y}\right) + O(\epsilon^2) \tag{32}$$

In the limit of y approaching zero,

$$\hat{G}_o = 4\pi\left\{\left(\frac{\partial I_b}{\partial y}\right)_0 y + \frac{1}{2}\left(\frac{\partial^2 I_b}{\partial y^2}\right)_0 y^2 + \frac{1}{6}\left(\frac{\partial^3 I_b}{\partial y^3}\right)_0 y^3 + O(y^4)\right\}$$
$$+ \epsilon 4\pi\left\{\left(\frac{\partial^2 I_b}{\partial y^2}\right)_0 - [1 + y + y^2 + O(y^3)]\left(\frac{\partial I_b}{\partial y}\right)_0\right\} + O(\epsilon^2) \tag{33}$$

Rewriting Eq. (33) in terms of the inner variable η,

$$\hat{G}_o = \epsilon^{1/2} 4\pi\left\{\left(\frac{\partial I_b}{\partial y}\right)_0 \eta\right\} + \epsilon\left\{-4\pi\left(\frac{\partial I_b}{\partial y}\right)_0\right.$$
$$\left. + 4\pi\left(\frac{\partial^2 I_b}{\partial y^2}\right)_0\left(1 + \frac{1}{2}\eta^2\right)\right\} + O(\epsilon^{3/2}) \tag{34}$$

The solution for $\hat{G}$ is uniformly valid if $\hat{G}_o$ as $y \to 0$ matches with $\hat{G}_i$ as $\eta \to \infty$ up to the order of terms retained in the series expansions. Therefore, the missing boundary conditions are obtained by matching the like terms of Eqs. (24) and (34) as follows.

$$\hat{G}_{i1} = 4\pi\left(\frac{\partial I_b}{\partial y}\right)_0 \eta \quad \text{as } \eta \to \infty \tag{35}$$

$$\hat{G}_{i2} = 4\pi\left[-\left(\frac{\partial I_b}{\partial y}\right)_0 + \left(\frac{\partial^2 I_b}{\partial y^2}\right)_0\left(1 + \frac{1}{2}\eta^2\right)\right] \quad \text{as } \eta \to \infty \tag{36}$$

Complete Solution

Equations (25) and (26) for first- and second-order inner terms, subject to boundary conditions Eq. (27), (28), (35), and (36), take on the following closed-form solutions.

$$\hat{G}_{i1} = 4\pi\left(\frac{\partial I_b}{\partial y}\right)_0\left[\frac{\gamma}{\gamma+1}\exp(-\eta) + \eta\right] \tag{37}$$

$$\hat{G}_{i2} = 4\pi\left(\frac{\partial I_b}{\partial y}\right)_0\left[-1 + \frac{1}{2}\frac{\gamma}{\gamma+1}\left(\eta + \frac{1+(\gamma+1)^2}{\gamma(\gamma+1)}\right)\exp(-\eta)\right]$$
$$+ 4\pi\left(\frac{\partial^2 I_b}{\partial y^2}\right)_0\left[1 + \frac{\eta^2}{2} - \frac{1}{(\gamma+1)}\exp(-\eta)\right] \tag{38}$$

Finally, the complete solution is

$$G_i = 4\pi I_w + \epsilon^{1/2}\hat{G}_{i1}(\eta) + \epsilon\hat{G}_{i2}(\eta) + O(\epsilon^{3/2}) \tag{39}$$

$$G_o = 4\pi I_b + \epsilon\hat{G}_{o2}(y) + O(\epsilon^2) \tag{40}$$

where $\hat{G}_{i1}$ is given by Eq. (37), $\hat{G}_{i2}$ by Eq. (38), and $\hat{G}_{o2}$ by Eq. (31).

IV. Results and Discussion

The problem at hand has one singularity which lies at the wall itself. Therefore, only one inner region and one outer region are required to describe its solution. The coordinate stretching of Eqs. (14) and (15) has the connotation of normalizing the independent variable in the inner region by the optical thickness. For a better comprehension, rewrite Eq. (12) as

$$1/\epsilon^{1/2} = [3/(1-\omega)]^{1/2}(1-\omega)\beta R$$

In the foregoing equation, $(1-\omega)\beta$ is the absorption coefficient and, therefore, $(1-\omega)\beta R$ is the optical thickness of the medium ($1/[(1-\omega)\beta]$ is the radiation mean free path); and the factor $(1-\omega)^{1/2}$ signifies that in the presence of scattering, the problem cannot be characterized by the optical thickness alone (the same factor also appears in the boundary condition). Thus, ϵ is a natural parameter of the problem.

The outer solution of Eq. (40) does not see the wall. The first term ($4\pi I_b$) describes the gas behavior in the limit of zero mean free path. In this limit, the curvature effect is not felt and the absorption and emission of thermal radiation are in local equilibrium at each point in the radiation field. The second term ($\epsilon\hat{G}_{o2}$) is the first-order correction to this behavior as the mean free path becomes finite but small. Thus, in the outer region

$$G - 4\pi I_b = O(\epsilon) \tag{41}$$

However, in the near wall region this ordering is not maintained, i.e., to $O(\epsilon)$, G and $4\pi I_b$ are not equal. In fact, according to the asymptotic solution of Eq. (39), in the inner region

$$G-4\pi I_w=0(\epsilon^{1/2}) \tag{42}$$

Now, the difference between G and $4\pi I_b$ is proportional to the divergence of heat flux [see Eq. (5)]. Thus, the magnitude of divergence of heat flux changes by $0(\epsilon^{1/2})$ as one approaches the wall from the outer region. This change in ordering clearly points to the existence of a radiation boundary layer for G in the limit of β becoming very large.

Equations (39) and (40) can be used to calculate the variation of heat flux in the inner and outer regions. These are

$$Q_i=[(1-\omega)/3]^{1/2}\epsilon^{1/2}\frac{\partial \hat{G}_{i1}}{\partial\eta}+\epsilon\frac{\partial \hat{G}_{i2}}{\partial\eta}+0(\epsilon^{3/2}) \tag{43}$$

$$Q_o=[(1-\omega)/3]^{1/2}\epsilon^{1/2}\left(4\pi\frac{\partial I_b}{\partial y}\right)+\epsilon^{3/2}\frac{\partial \hat{G}_{o2}}{\partial y}+0(\epsilon^2) \tag{44}$$

If no singular perturbation analysis were carried out, the calculated wall heat flux would correspond to the lead term of Eq. (44). Let this be called Q_{Dw} where the subscript D stands for the diffusion approximation.

$$Q_{Dw}=[(1-\omega)/3]^{1/2}\epsilon^{1/2}4\pi\left(\frac{\partial I_b}{\partial y}\right)_0 \tag{45}$$

An alternative form of the foregoing equation is

$$Q_{Dw}=-\frac{4\pi}{3\beta}\left(\frac{\partial I_b}{\partial r}\right)_0 \tag{46}$$

The correct solution for wall heat flux, on the other hand, should be calculated from Eq. (43). Up to $0(\epsilon^{1/2})$, this is

$$Q_w=\frac{1}{\gamma+1}Q_{Dw}+0(\epsilon) \tag{47}$$

which means that the diffusion approximation predicts a heat flux too high by a factor of $(\gamma+1)$. Up to $0(\epsilon)$, the wall heat flux is

$$Q_w=\frac{1}{\gamma+1}Q_{Dw}\left\{1+\epsilon^{1/2}\left[-\frac{(\gamma+2)}{2(\gamma+1)}+\frac{(\partial^2 I_b/\partial y^2)_0}{(\partial I_b/\partial y)_0}\right]\right\}+0(\epsilon^{3/2}) \tag{48}$$

For calculating heat flux in the entire radiation field, Eqs. (43) and (44) should be used. It is interesting to note that in the diffusion approximation, the only manner in which scattering influences heat flux is through the extinction coefficient, i.e., the effect of in-scattering of radiation on heat flux does not appear in the diffusion approximation. In the correct asymptotic solution, however, the in-scattering effects enter through the appearance of γ in the solution.

Finally, according to the first-order outer solution, Eq. (40), the value for G at any point in the radiation field is the black body emissive power $(\equiv 4\pi I_b)$. Consequently, $G_{o1}=4\pi I_w$ at the wall. According to the correct solution, however, the difference between G and $4\pi I_w$ in the wall region is of $0(\epsilon^{1/2})$, Eq. (42). Precisely, at the wall

$$G_w-4\pi I_w=4\pi\left(\frac{2\gamma+1}{\gamma+1}\right)\left(\frac{\partial I_b}{\partial y}\right)_0\epsilon^{1/2}+0(\epsilon)$$

This difference between gas emissive power close to the wall and $4\pi I_w$ has been termed a jump in "emissive power at wall" by Deissler.[2]

V. Conclusions

The equation governing the transport of incident radiation becomes mathematically singular as the optical thickness becomes large. This singularity is an attribute of a small parameter ϵ, inversely proportional to the square of optical thickness, multiplying the highest derivative of the differential equation. It is found that in the optically thick limit, away from the wall the radiation field is in near equilibrium, inasmuch as up to $0(\epsilon)$ the rates of absorption and emission of radiation are equal. In the near-wall region, however, the absorption and emission rates are different by $0(\epsilon^{1/2})$. In other words, the magnitude of the divergence of radiative heat flux changes by $0(\epsilon^{1/2})$ as one approaches the wall from the far region. Thus, there exists a radiation boundary layer for the incident radiation in the limit of optical thickness becoming very large.

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